Evolution of Social preferences. The comparison of media influenced and uninfluenced dynamics

by

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A final project submitted in partial fulfillment
of the requirements for the degree of
Master of Science in Economics
Department of Economics
New York University Abu Dhabi
June, 2024

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Abstract

In this final project, we present a model of social influence among actors, each possessing varying degrees of credibility in the eyes of others. The central focus of this model is the concept of preference relations over a given set, interpreted as a set of policies. Actors engage in iterative discussions to resolve their disagreements about the ranking of policy pairs, eventually reaching a final conclusion under certain equilibrium conditions. Additionally, we explore the impact of media perturbations, where specific narratives transmitted to agents alter the equilibrium preferences formed, the time it takes for them to converge, as well as diversities and manipulabilities of preference profiles along the equilibrium path. Our findings reveal that within our behavioral mechanism and group structure, preferences of representatives from larger social groups are more susceptible to media manipulation than those from smaller groups. Furthermore, empirically we observe that under most of the network structures, each successive step toward equilibrium, if it exists, weakly reduces diversity (without necessarily eliminating it), thereby decreasing the manipulability of agents' preferences by the media. Consequently, in such cases as agents' preferences approach equilibrium, their susceptibility to media influence diminishes.

Key words: Social choice theory, voters, preferences, candidates, equilibrium, preference update mechanism, influence, diversity of preferences, manipulability of preferences.

JEL code: D71, D72, D85

1 Introduction

Social systems consist of numerous agents who interact by exchanging specific information and forming their attitudes toward various phenomena based on these exchanges. The dynamics of this process, where individuals shape their opinions based on others' views—or even on what they believe others think—can lead to diverse outcomes, such as social consensus, polarization, or an infinite updating process, which we formally call the non-existence of equilibrium. Crucially, the notion that agents do not base their decisions solely on the opinions of all societal representatives, but rather rely to some extent on what the media conveys about others' preferences, underpins our approach of incorporating media exposure during the dynamic process of preference updates.

As stated by Hegselmann and Krause (2002), "in general, an agent will neither simply share nor strictly disregard the opinion of any other agent, but will take into account the opinions of others to a certain extent in forming his own opinion. This can be modeled by different weights which any of the agents puts on the opinions of all the other agents." This idea justifies the use of weights in the preference update mechanism in our model.

A natural question arises: do agents shape collective preferences, or do collective preferences shape individual ones? From an economic standpoint, it is typically the agents' preferences that shape the collective's. However, our model also considers the more social scientific perspective, acknowledging the inverse effect: collective outcomes likewise shape individual preferences, which then influence social preferences again, creating an iterative cycle. Despite this potentially infinite cycle, models that discuss such iterative updates can still reach a state where no further updates occur, known as equilibrium.

In this final project, we emphasize the dynamics of this process, focusing on the fluid nature of preferences. We make the reasonable assumption that agents need not be strictly rational and thus do not impose strong assumptions such as transitivity of preferences. Instead, we concentrate on the dynamics of the updating process and the interplay between group size, structure, diversity, and manipulability of preferences.

Following Coleman (1990)'s classification of collective behavior phenomena, which includes decision making, consumer choices, election voting, social impact, and opinion formation, we focus specifically on the latter two. To formally capture these phenomena, our model includes two main components: the notion of group structure, which defines the degree of influence among agents, and a preference update mechanism that explains how new preferences are formed under the given group structure.

In our setting, we assume a group of voters, each with a preference relation over policies. As Nowak and Lewenstein (1996) have shown in their analysis of Latane and Nowak (1994)'s work, people tend to adopt extreme attitudes on issues they find subjectively important. Therefore, it is reasonable to use a binary representation of preferences when ranking two candidates. Our study scenario is as follows:

Each discrete point in time, voters meet with others whose positions are important to them. They discuss each pair of policies, and at the end of the period, the influenced voter updates their preferences, accounting for the credibility they attribute to others. This process repeats until no one changes their preferences, reaching equilibrium.

We further consider the perturbation of this system by media shocks. During a discussion, media (e.g., TV or radio with advertising or propaganda) captures part of the voters' attention, transmitting a specific narrative about policies. In the next period, voters resume their discussions without media influence until equilibrium is reached.

Our findings indicate that, under most of the group structures and initial preferences shared by the agents, the preference update mechanism reduces diversity without necessarily completely eliminating it. The effect of media shocks varies with group size: preferences in smaller groups are more stable and less influenced by media, while in larger groups, the discussion becomes noisier, amplifying the media's impact and making preferences highly manipulable. However, for most of the group structures and initial preference profiles, diversity in preferences at equilibrium is weakly reduced regardless of group size.

Our aim is to construct a social influence model that formalizes this process and discusses its implications. In the following chapters, we present a literature review, classify existing approaches to such questions, and then introduce our model with theorems, proofs, and examples.

To summarize, our model demonstrates that the manipulability of preferences by media increases with group size, *ceteris paribus.* Additionally, depending on the group structure manipulability decreases as diversity in preferences reduces, which typically occurs along the equilibrium path, given all other parameters remain constant.

2 Literature Review

According to Boudourides and Scarlatos (2004) there are conceptually four different ways to mathematically formalise models of Social influence. They are

Game theory Bramoulle (2002), Durlauf and Young (2001), Macy (1990), and Young (1998, 2005)
In this framework, agents are assumed to be rational, with this rationality being common knowledge. Consequently, agents act strategically, typically to maximize their utility. Therefore, the preference update mechanism is determined solely by the agents' best responses and strategic interactions.

Statistical mechaniscs and computer simulations Latane (1981), Lewenstein et al. (1992), and Nowak et al. (1990)
This approach draws insights from physical interactions between objects or biological interactions between various species, using such mathematical objects as cellular automata, and applies the principles of these interactions to agents, conceptualizing preference formation in this man-

ner. In other words, insights from natural science are applied to study social science concepts. As noted by Nowak and Lewenstein (1996), computer simulation is employed in this approach as it provides insights and intuition on social processes, albeit at the cost of trivializing our understanding by offering a mirage of precise description.

Interacting particle systems Durrett (1988, 1999), Griffeath (1998), Liggett (1985, 1997, 1999), Nakamaru and Levin (2004), and Spitzer (1971)
Initially, interacting particle system models were developed to describe and analyze stochastic models where evolution occurs over time, such as the spread of a disease, tumor growth, and later, the dynamics of behavioral systems. In this context, the statistical framework of a stochastic process is employed to formalize the change in voters' preferences over time, with voters being embedded in the structure of a connected graph.

Social network analysis Abelson (1964), Crès and Tvede (2022), French (1956), Friedkin (1999, 2001), Friedkin and Johnsen (1990, 1999), Harary (1959), Kobayashi (2001), and Stokman and Van den Bos (1992)
Under models of this type, agents constitute a social influence network structure, typically represented by a weighted graph that connects voters according to their influence relationships. The strength of such influence is represented by values, such as the weights of the edges in the social influence network graph. Besides the structure of the social network, a preference mechanism should be presented on that structure, allowing for the modeling of each agent's preference update.

In practice, while building models and implementing simulations of their performance, the methodologies described above are applied simultaneously, with one approach often being emphasized over others. Despite differences in these approaches, according to Boudourides and Scarlatos (2004), all the models make similar assumptions, specifically,

1. What is the set of agents, that are taking part in the process of social influence one on another
2. What is the set of possible outcomes, that arise as a result of such social interaction (influence) process
3. What are the rules of interaction, that define how specifically the influence mechanism is working, and how the outcomes are shaped.

We will address these questions in the sections below.

3 Model

3.1 Model Set up and an Equilibrium

As discussed in the Literature Review Section, four main approaches exist for constructing models of Social Influence. Given that agents in our setting possess varying degrees of influence, follow leaders, and tend to establish strong connections with representatives of their families, friends, coworkers, and neighbors, we have chosen the Social Network Analysis approach to construct a societal model. This decision is justified by the absence of an assumption that agents behave strategically; instead, we posit the existence of a behavioral mechanism guiding their decisions.

Additionally, another reason for specifically choosing the Social Network Theory approach is our objective to study the implications of preference update mechanisms and group structures on formed preferences. Adhering to Occam's razor, we refrain from introducing unnecessary complexity by excluding the concept of uncertainty from our model. Thus, we remain within the paradigm of Social Network Analysis, alongside employing computer simulations to illustrate our ideas. Let us now delve into the formalization of the model.

Let $N = \{v_1, ...v_n\}$ be the set of voters, and $K = \{k_1, ...k_L\}$ be the set of policies. All voters' preferences at a specific point of time t constitute a preference profile $(R_1^{(t)}, ..., R_n^{(t)})$. We will also denote $R_{-s}^{(t)} = (R_1^{(t)}, ..., R_{s-1}^{(t)}, R_{s+1}^{(t)}, ..., R_n^{(t)})$ to be the preference profile, that includes all the preferences of the agents in N except for preference of agent $s \in N$.

Further, to formalise that position of one of the agents influences the position of the others in a certain way, there exist an influence matrix $W_{N \cdot N} = [w_{sr}]$ such that $\forall s, r \in N\ w_{sr} \in \mathbb{R}_+$ and on top of that a technical normalisation condition, that $\forall s \in N\ \sum_{r=1}^{N} w_{sr} = 1$ is satisfied. The weight w_{sr} is interpreted as credibility (importance) of position of agent r for agent s.
For each voter $s \in N$ we will use the s^{th} row of matrix W as the vector of weights (credibilities) that agent s puts into words of each other agent.

Denote $\mathbb{M}_N$ the set of all possible matrices W of size $|N| \times |N|$, that an agent can have. Before defining the preference update mechanism note, that preference $R_s^{(t)} \subseteq K^2$ can be represented in matrix form as follows:

$$[R_s^{(t)}]_{ij} = \begin{cases} 1 \text{ if } (k_i, k_j) \in R_s^{(t)} \\ 0 \text{ otherwise} \end{cases} \quad \forall i, j \in N$$

Definition 3.1 (Preference update mechanism). For a preference profile define a functional $f : 2^{K^2} \times (2^{K^2})^{|N-1|} \times \mathbb{M}_{\mathbb{N}} \longrightarrow 2^{K^2}$ to be a preference update mechanism.[1]

In our model we take the mechanism $f(\cdot)$ for any agent $s \in N$ to be $f(R_s^{(t)}, R_{-s}^{(t)}, W^{(t)})$

$$\text{such that } [f(R_s^{(t)}, R_{-s}^{(t)}, W^{(t)})]_{ij} = pos(\sum_{r \in N} w_{sr} \cdot [R_r^{(t)}]_{ij} - \frac{1}{2} \sum_{r \in N} w_{sr})\ \forall i, j \in K,$$

[1]Note that f is a function with a changing number of variables: when there is no media intervention $|N| = n$, whereas when there is one, the media is considered to be an auxiliary agent and then $|N| = n + 1$

where $pos : \mathbb{R} \longrightarrow \{0;1\}$ such that $pos(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ otherwise} \end{cases}$

and $W^{(t)} = \begin{cases} W \text{ if there is no exposure to the media shock at time } t \\ W^* \text{ if there is an exposure to the media shock at time } t \end{cases}$

When the media shock is added to our model, we formalise it by perturbing matrix W to W^*. Specifically, we add an auxiliary $n+1^{\text{st}}$ agent (media), that has the following credibility weights for each of the voters $w_{n+1,s} = 0 \quad \forall s \in N$, $w_{s,n+1} = \frac{n-1}{2n+2} \quad \forall s \in N$ and $w_{n+1,n+1} = 0$, with all other weights remaining the same as in matrix W. This means, that the voters do not influence the media, but each voter is influenced by the media, and magnitude of such influence is increasing when the group size becomes larger, and is determined by the weight $w_{s,n+1}$.

When the media enters discussion, the natural question that arises is how much weight to assign it. Let $x(n)$ be normalised influence (weight) of media to each member of the group of n agents. Where under "normalised" we mean, that $x(n) = \frac{w_{s,n+1}}{\sum\limits_{r=1}^{n} w_{sr} + w_{s,n+1}}$. Assuming that initial matrix W is normalised (i.e. $\sum\limits_{r=1}^{n} w_{sr} = 1 \; \forall s \in N$) we have, that $x(n) = \frac{w_{s,n+1}}{1+w_{s,n+1}} \Leftrightarrow w_{s,n+1} = \frac{x(n)}{1-x(n)}$.

To capture the idea, that the influence of media should increase with n, we require the sequence $x(n)$ to be increasing in n.

Therefore assume $x(1) = 0$ (media does not have an effect on a group of one voter). Further note, that if $x(n) > 0.5$ for some n it would mean, that media becomes a dictator in the Arowian sense (i.e. whatever the preference relation it communicates, everyone will accept it), and thus the system will end up in a unanimous equilibrium on the next iteration after the media shock. This case is not of particular interest, so it is reasonable to bound sequence $x(n)$ with 0.5 from above, that is we need to take $x(n)$ such that $lim_{n\to\infty} x(n) = 0.5$ and $x(1) = 0$ with $x(n)$ being increasing. One can make different assumptions on which sequence to take depending on its speed of convergence. Throughout this project, we will take $x(n) = \frac{4n}{3n+1} - 1$ and consequently,

$w_{s,n+1} = \frac{n-1}{2n+2}$. Our motivation to take such a sequence is that we want to study the media effect on large samples by taking smaller samples, to ensure easier tractability, simplicity and clarity of the model. So we will transfer properties of the large groups to the smaller groups, and use the latter as the sandboxes.

Definition 3.2 (Equilibrium). A preference profile $(R_1^{(t)}, ..., R_n^{(t)})$ is an equilibrium if and only if $\forall s \in N \; f(R_s^{(t)}, R_{-s}^{(t)}, W^{(t)}) = R_s^{(t)}$

3.2 Further Definitions, Propositions and Theorems

Definition 3.3 (Notion of Kemeny distance). A distance between preference relations R_s and R_r with $r, s \in N$ is defined as $d(R_s, R_r) = |(a,b) \in K^2 : (a,b) \in R_s \cap R_r^c \; or (a,b) \in R_r \cap R_s^c \; |$

Definition 3.4 (Diversity of a preference profile). For a preference profile $(R_1, ..., R_n)$ we define a notion of diversity as $div(R_1, ..., R_n) = \frac{1}{n^2} \sum_{s=1}^{n} \sum_{r=1}^{n} d(R_s, R_r)$

Note that since Kemeny distance is a distance in a metric sense, it satisfies the Symmetry Axiom. Therefore the diversity of preference profile can be equivalently calculated (or be defined as): $div(R_1, ..., R_n) = \frac{2}{n^2} \sum_{s=1}^{n} \sum_{r=s+1}^{n} d(R_s, R_r)$

The interpretation of the diversity of the preference profile is the average Kemeny distance between two preferences in the preference profile.

Further we want to derive the notion of manipulability of a preference profile. This notion will quantify what is the largest possible impact, that the media can have on a preference profile within one time period, and we will define it as the maximal possible distance between the preference profile, that will be formed on the next iteration without agents being exposed to the media shock and preference profile on the next iteration given they are exposed to the media shock.

Before doing that define the notion of degree of manipulability of a specific media signal under the given preference profile and group structure:

Definition 3.5 (Notion of manipulability of media signal for the given preference profile and the group structure). Given the influence matrix W , the influence matrix perturbed by the media W^* and the preference profile $(R_1, ..., R_n)$ define the degree of the manipulability of a media signal $R_{med} \in 2^{K^2}$ as

$manip(R_{med}) = \frac{1}{n} \sum_{s=1}^{n} d(f(R_s, R_{-s}, W), f(R_s, R^*_{-s}, W^*)),$

where $R^*_{-s} = (R_1, ..., R_{s-1}, R_{s+1}, ...R_n, R_{med})$. In other words, the degree of manipulability of the media signal for the specific preference profile under the given group structure is the average Kemeny distance between the preferences in the preference profile on the next stage under no media intervention, and the preferences in the preference profile under the media intervention with signal R_{med}.

Then one can define the notion of manipulability of the preference profile as the maximal possible manipulability, that the media can have:

Definition 3.6 (Notion of manipulability of preference profile). Given influence matrix W and influence matrix perturbed by the media W^* define the degree of the manipulability of a preference profile $(R_1, ..., R_n)$ as

$manip(R_1, ..., R_n) = \max_{R_{med} \in 2^{K^2}} manip(R_{med})$

Which by the definition of manipulability of the media signal is equal to

$\max_{R_{med} \in 2^{K^2}} \frac{1}{n} \sum_{s=1}^{n} d(f(R_s, R_{-s}, W), f(R_s, R^*_{-s}, W^*))$. In other words, the degree of manipulability of the preference profile is the maximal average Kemeny distance between the preference relations in the preference profile on the next stage under no media intervention, and the preference relations in the preference profile with the media intervention.

We will further call a "spoiler" the corresponding media signal (note that there can be several of those), that changes the largest possible number of rankings in one period: $R_{med} \in 2^{k^2}$ such that

$R_{med} \in \underset{R^*_m \in 2^{K^2}}{argmax} \frac{1}{n} \sum_{s=1}^{n} d(f(R_s, R_{-s}, W), f(R_s, R^*_{-s}, W^*))$. Note that $R^*_m \in 2^{K^2}$ is the last component of preference profile R^*_{-s}.

Proposition 3.1 (Representation of the outcome of behavioural mechanism $f(\cdot)$ as the outcome of majority voting for some artificial preference profile). *Let W be the n by n influence matrix such that $\forall s, k \in N$ $w_{ss} \neq 0$ and $w_{sk} \in \mathbb{Q}_+$ and f-the preference update mechanism $f(\cdot)$ defined above. Then $\forall s \in N$ $f(R_s^{(t)}, R_{-s}^{(t)}, W^{(t)})$ is the outcome of majority voting for the preference profile* $I = (\underbrace{R_1^{(t)}, \ldots, R_1^{(t)}}_{\frac{p_1 \cdot LCM(q_1,\ldots,q_n)}{q_1}}, \ldots, \underbrace{R_k^{(t)}, \ldots, R_k^{(t)}}_{\frac{p_k \cdot LCM(q_1,\ldots,q_n)}{q_k}}, \ldots, \underbrace{R_n^{(t)}, \ldots, R_n^{(t)}}_{\frac{p_n \cdot LCM(q_1,\ldots,q_n)}{q_n}})$

Proof. First note, that $\forall s, k \in N$ $w_{sk} \in \mathbb{Q}_+ \overset{\text{def } \mathbb{Q}_+}{\Leftrightarrow} \exists p_k, q_k \in \mathbb{Z} : GCD(p_k, q_k) = 1 | w_{sk} = \frac{p_k}{q_k}$.

Then we can rewrite our preference update mechanism as

$$\forall s \in N\ \forall i, j \in K\ [f(R_s^{(t)}, R_{-s}^{(t)}, W^{(t)})]_{ij} \overset{\text{def } f(\cdot)}{=} pos(\sum_{k \in N} w_{sk} \cdot [R_k^{(t)}]_{ij} - \frac{1}{2} \sum_{k \in N} w_{sk}) \overset{def \mathbb{Q}_+}{=}$$

$$= pos(\sum_{k \in N} \frac{p_k}{q_k} \cdot [R_k^{(t)}]_{ij} - \frac{1}{2} \sum_{k \in N} \frac{p_k}{q_k}) = pos(\sum_{k \in N} \frac{p_k}{q_k} \cdot ([R_k^{(t)}]_{ij} - \frac{1}{2}) = pos(\sum_{k \in N} \frac{p_k \cdot \frac{LCM(q_1,\ldots,q_n)}{q_k}}{LCM(q_1,\ldots,q_n)} \cdot ([R_k^{(t)}]_{ij} - \frac{1}{2})) = pos(\sum_{k \in N} \frac{p_k}{q_k} \cdot ([R_k^{(t)}]_{ij} - \frac{1}{2})) = pos(\frac{1}{LCM(q_1,\ldots q_n)} \sum_{k \in N} p_k \cdot \frac{LCM(q_1,\ldots,q_n)}{q_k} \cdot ([R_k^{(t)}]_{ij} - \frac{1}{2})) \overset{pos(\cdot) \text{is homogeneous of degree 0}}{=}$$

$$= pos(\sum_{k \in N} p_k \cdot \frac{LCM(q_1,\ldots,q_n)}{q_k} \cdot ([R_k^{(t)}]_{ij} - \frac{1}{2})) \overset{\text{def } pos(\cdot)}{=} \begin{cases} 1 \text{ if } \sum_{k \in N} p_k \cdot \frac{LCM(q_1,\ldots,q_n)}{q_k} \cdot ([R_k^{(t)}]_{ij} > \frac{1}{2} \sum_{k \in N} p_k \cdot \frac{LCM(q_1,\ldots,q_n)}{q_k} \\ 0 \text{ otherwise} \end{cases},$$

which is the preference relation resulting from a majority voting, where each binary relation $R_k^{(t)}$ from preference profile $(R_1^{(t)}, \ldots R_n^{(t)})$ is shared $p_k \cdot \frac{LCM(q_1,\ldots,q_n)}{q_k}$ times.

Therefore construct preference profile $I = (\underbrace{R_1^{(t)}, \ldots, R_1^{(t)}}_{\frac{p_1 \cdot \text{LCM}(q_1,\ldots,q_n)}{q_1}}, \ldots, \underbrace{R_k^{(t)}, \ldots, R_k^{(t)}}_{\frac{p_k \cdot \text{LCM}(q_1,\ldots,q_n)}{q_k}}, \ldots, \underbrace{R_n^{(t)}, \ldots, R_n^{(t)}}_{\frac{p_n \cdot \text{LCM}(q_1,\ldots,q_n)}{q_n}})$. Note, that it contains $M = \sum_{i=1}^{n} \frac{p_i \cdot LCM(q_1,\ldots,q_n)}{q_i}$ preference relations in it. We have just shown above, that following the behavioural mechanism $f(\cdot)$ for agent $s \in N$ will be equivalent to following the outcome of a majority voting for such a preference profile I.

□

Theorem 3.2 (Preservation of completeness property of binary relation by preference update mechanism). *Let W be the influence matrix such that $\forall s, k \in N w_{ss} \neq 0$ and $w_{sk} \in \mathbb{Q}_+$ and f-the preference update mechanism $f(\cdot)$ defined above. Then if $\forall s \in N$ R_s is complete, then $\forall s \in N$ $f(R_s, R_{-s}, W)$ is complete.*

Proof. First apply the previous observation to claim, that $\forall s \in N$ $f(R_s, R_{-s}, W)$ is the outcome of majority voting for some preference profile $I = (\underbrace{R_1, \dots, R_1}_{\frac{p_1 \cdot \text{LCM}(q_1,\dots,q_n)}{q_1}}, \dots, \underbrace{R_k, \dots, R_k}_{\frac{p_k \cdot \text{LCM}(q_1,\dots,q_n)}{q_k}}, \dots, \underbrace{R_n, \dots, R_n}_{\frac{p_n \cdot \text{LCM}(q_1,\dots,q_n)}{q_n}})$. Since the preference profile I contains only preferences R_1 to R_n in different quantities, and by assumption of the theorem all of them are complete, preference profile I consists of complete preferences.

Assume, that such majority voting preference $R_{majority}$ relation is not complete $\overset{\text{def complete}}{\Leftrightarrow}$
$\exists a, b \in K$ such that $(a, b) \notin R_{\text{majority}}$ and $(b, a) \notin R_{\text{majority}}$ $\overset{\text{def majority voting}}{\Leftrightarrow}$
more than $\frac{|I|}{2}$ preferences in the preference profile I rank $(a, b) \notin R$
and more that $\frac{|I|}{2}$ preferences in the preference profile I rank $(b, a) \notin R$ $\overset{\text{Pigeonhole principle}}{\Longrightarrow}$
$\exists R \in I$ such that $(a, b) \notin R$ and $(b, a) \notin R$ $\overset{\text{def complete}}{\Leftrightarrow}$ $\exists R \in I |$ R is not complete $\Leftrightarrow$
$\Leftrightarrow$ contradiction with $\forall R \in I$ R being complete. This concludes the proof about preservation of completeness through our behavioural mechanism $f(\cdot)$ □

Observation 3.3 (Non-preservation of transitivity property of binary relation by preference update mechanism). *Let W be the influence matrix such that $\forall s, r \in N$ $w_{ss} \neq 0$ and $w_{sr} \in \mathbb{Q}_+$ and f-the preference update mechanism $f(\cdot)$ defined above. Then if $\forall s \in N$ R_s is transitive, then it is not necessarily holds, that $\forall s \in N$ $f(R_s, R_{-s}, W)$ is transitive.*

Proof. A counterexample will be a sufficient proof in this case. Take one based on Condorcet paradox. Let $W = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$ and $R_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ $R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Note that each of R_1, R_2, R_3 is transitive. However, $f(R_1, (R_2, R_3), W) = f(R_2, (R_1, R_3), W) = f(R_3, (R_1, R_2), W) =$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, which is not transitive. □

Theorem 3.4 (Sufficient condition of equilibrium existence). *Let W be the lower triangular influence matrix (i.e. $\forall s, k : k > s \in N \ w_{sk} = 0$), that satisfies regularity condition $\sum_{k=1}^{n} w_{sk} = 1 \ \ \forall k \in N$ and f-the preference update mechanism $f(\cdot)$ defined in the Model Set-up. Then for each initial preference profile $(R_1^{(0)}, ..., R_n^{(0)})$ there exist a finite $t \in \mathbb{N} \cup 0$ such that the preference profile at time t $(R_1^{(t)}, ..., R_n^{(t)})$ will be an equilibrium preference profile.*

Proof. Let $W = \begin{pmatrix} w_{11} & 0 & 0 & \cdots & 0 \\ w_{21} & w_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & w_{n3} & \cdots & w_{nn} \end{pmatrix}$ be the given lower triangular influence matrix such that $\sum_{k=1}^{n} w_{sk} = 1 \ \ \forall s \in N$.

We will show by induction, that it will take at most $n-1$ steps to reach the equilibrium regardless of the preference profile we start with.

Base. Let $n = 1$. Then $\forall i, j \in K \ [R_1^{(1)}]_{ij} = [f(R_1^{(0)}, R_{-1}^{(0)}, W_{1 \cdot 1})]_{ij} = pos(w_{11} \cdot [R_1^{(0)}]_{ij} - \frac{1}{2} w_{11}) = pos(1 \cdot [R_1^{(0)}]_{ij} - \frac{1}{2} \cdot 1) = pos([R_1^{(0)}]_{ij} - \frac{1}{2}) = \begin{cases} pos(\frac{1}{2}) \text{ if } [R_1^{(0)}]_{ij} = 1 \\ pos(-\frac{1}{2}) \text{ if } [R_1^{(0)}]_{ij} = 0 \end{cases} = \begin{cases} 1 \text{ if } [R_1^{(0)}]_{ij} = 1 \\ 0 \text{ if } [R_1^{(0)}]_{ij} = 0 \end{cases} = [R_1^{(0)}]_{ij}$.

So that $\forall i, j \in K \ [f(R_1^{(0)}, R_{-1}^{(0)}, W_{1 \cdot 1})]_{ij} = [R_1^{(0)}]_{ij} \Leftrightarrow f(R_1^{(0)}, R_{-1}^{(0)}, W_{1 \cdot 1}) = R_1^{(0)}$, which by definition of Equilibrium means, that the initial preference profile $(R_1^{(0)})$ is the equilibrium preference profile. Note that it took no more that 0 steps ($n - 1$ steps for $n = 1$) to reach it.

Assumption Assume it takes exactly m steps with $m \leq (n-1) - 1 = n-2$ for $n-1$ agents to reach the equilibrium. Then: $\forall t \geq m \ \forall s \in \{1, .., n-1\} \ f(R_s^{(t)}, R_{-s}^{(t)}, W_{n \cdot n}) = R_s^{(t)} = R_s^{(e)} \ \forall s \in \{1, ..., n-1\}$.
We will denote $(R_1^{(e)}, ..., R_{n-1}^{(e)})$ the equilibrium preference profile for agents 1 to $n - 1$, that by assumption of induction exists and is reached in $m \leq n - 2$ steps.

Step Then for agent n at time $m+1$ we have: $\forall i, j \in K\ [R_n^{(m+1)}]_{ij} = [f(R_n^{(m)}, R_{-n}^{(m)}, W_{n \cdot n})]_{ij} \overset{\text{def} f(\cdot)}{=}$

$$= pos(\sum_{k=1}^{n-1} w_{n,k} \cdot [R_k^{(e)}]_{ij} + w_{nn} \cdot [R_n^{(m)}]_{ij}) = \begin{cases} 1 \text{ if } \sum_{k=1}^{n-1} w_{n,k} \cdot [R_k^{(e)}]_{ij} + w_{nn} \cdot [R_n^{(m)}]_{ij} > \frac{1}{2} \\ 0 \text{ otherwise} \end{cases}$$

Then we need to consider 4 possible cases:

Case 1 $[R_n^{(m)}]_{ij} = 0$ and $\sum_{k=1}^{n-1} w_{n,k} \cdot [R_k^{(e)}]_{ij} + w_{nn} \cdot [R_n^{(m)}]_{ij} > \frac{1}{2}$. Then $\sum_{k=1}^{n-1} w_{n,k} \cdot [R_k^{(e)}]_{ij} + w_{nn} \cdot 0 > \frac{1}{2}$ $\forall t \geq m+1 \Rightarrow [R_n^{(t)}]_{ij} = 1\ \forall t \geq m+1$, so equilibrium for agent n is reached in one additional $m+1^{\text{st}}$ step.

Case 2 $[R_n^{(m)}]_{ij} = 1$ and $\sum_{k=1}^{n-1} w_{n,k} \cdot [R_k^{(e)}]_{ij} + w_{nn} \cdot [R_n^{(m)}]_{ij} > \frac{1}{2}$. Then $\sum_{k=1}^{n-1} w_{n,k} \cdot [R_k^{(e)}]_{ij} + w_{nn} \cdot 1 > \frac{1}{2}$ $\forall t \geq m+1 \Rightarrow [R_n^{(t)}]_{ij} = 1\ \forall t \geq m+1$, so equilibrium for agent n was already reached on or before time point m.

Case 3 $[R_n^{(m)}]_{ij} = 0$ and $\sum_{k=1}^{n-1} w_{n,k} \cdot [R_k^{(e)}]_{ij} + w_{nn} \cdot [R_n^{(m)}]_{ij} \leq \frac{1}{2}$. Then $\sum_{k=1}^{n-1} w_{n,k} \cdot [R_k^{(e)}]_{ij} + w_{nn} \cdot 0 \leq \frac{1}{2}$ $\forall t \geq m+1 \Rightarrow [R_n^{(t)}]_{ij} = 0\ \forall t \geq m+1$, so equilibrium for agent n is reached in one additional $m+1^{\text{st}}$ step.

Case 4 $[R_n^{(m)}]_{ij} = 1$ and $\sum_{k=1}^{n-1} w_{n,k} \cdot [R_k^{(e)}]_{ij} + w_{nn} \cdot [R_n^{(m)}]_{ij} < \frac{1}{2}$. Then $\sum_{k=1}^{n-1} w_{n,k} \cdot [R_k^{(e)}]_{ij} + w_{nn} \cdot 0 \leq \frac{1}{2}$ $\forall t \geq m+1 \Rightarrow [R_n^{(t)}]_{ij} = 0\ \forall t \geq m+1$, so equilibrium for agent n was already reached on or before time point m.

Therefore since given that $n-1$ agents reached their equilibrium preferences in m steps, it will take n^{th} agent at most $m+1$ steps to reach the equilibrium. Which means, that it takes at most $n-1$ steps for the whole system of n agents to reach the equilibrium preference profile for a group structure formalised by the lower triangular matrix. □

Proposition 3.5 (Representation of a preference update mechanism for extended social groups with the same group structure as of the original groups). *Let $N = \{v_1, \dots v_n\}$ be the original set of agents and $K = \{k_1, \dots k_L\}$ be the set of policies agents have their preferences $(R_1, \dots R_n)$ over. Further let $kN = \{v_1, \dots, v_{kn}\}$ be the extended set of agents that have preferences $(R_1, \dots, R_{kn})$ over the same set K with $R_{mn+s} = R_s \quad \forall m \in \{0; \dots k-1\}$ and $s \in \{1, \dots n\}$. Let W be* *an $n \times n$ influence matrix of the original group of agents, let W^* be an $(n+1) \times (n+1)$ influence matrix (which is matrix W perturbed by the media) of the original group of agents, let $W' = \mathbf{I}_k \otimes W$ be the $(kn) \times (kn)$ influence matrix of the extended group of agents kN, with $k \in \mathbb{N}$ (that preserves the same structure as W), and let W'^* be the corresponding $(kn+1) \times (kn+1)$ perturbation of the influence matrix W' of the extended society by the media, and let $f(\cdot)$ be a preference update mechanism defined above. Then:*

1. $f(R_{mn+s}, R'_{-(mn+s)}, W') = f(R_s, R_{-s}, W) \quad$ *for* $m \in \{0, \ldots, k-1\};\ s \in \{1, \ldots, n\}$

2. $f(R_{mn+s}, R'^*_{-(mn+s)}, W'^*) = f(R_s, R^*_{-s}, W^*) \quad$ *for* $m \in \{0, \ldots, k-1\};\ s \in \{1, \ldots, n\}$ *,where*

$R'_{-(mn+s)} = (R_1, \ldots, R_{mn+s-1}, R_{mn+s+1}, \ldots R_{nk})$

$R'^*_{-(mn+s)} = (R_1, \ldots, R_{mn+s-1}, R_{mn+s+1}, \ldots R_{nk}, R_{med})$

Proof. $\forall i, j \in K \quad \left[f(R_{mn+s}, R'_{-(mn+s)}, W')\right]_{ij} \overset{\text{def} f(\cdot)}{=} pos\left(\sum_{r \in kN} w_{mn+s,r} \cdot [R_r]_{ij} - \frac{1}{2} \sum_{r \in kN} w_{mn+s,r}\right) =$

$$\overset{\text{by construction of } W' \ w_{mn+s,r}=0 \text{ for } 1 \le r \le mn \text{ or } r>(m+1)n}{=} pos\left(\sum_{r \in \{mn+1, \ldots (m+1)n\}} w_{mn+s,r} [R_r]_{ij} - \frac{1}{2} \sum_{r \in \{mn+1, \ldots (m+1)n\}} w_{mn+s,r}\right) =$$

$$\overset{\text{by construction of } W' \ w_{mn+s,mn+r}=w_{s,r} \ \forall s,r \in \{1,\ldots,n\}}{=} pos\left(\sum_{r \in N} w_{sr} [R_r]_{ij} - \frac{1}{2} \sum_{r \in N} W_{sr}\right) \overset{\text{def} f(\cdot)}{=} [f(R_s, R_{-s}, W)]_{ij}.$$

So that we have proven matrix componentwise the assertion (1), i.e. that $f(R_{mn+s}, R'_{-(mn+s)}, W') = f(R_s, R_{-s}, W)$.

To prove assertion (2) add an auxiliary $nk+1^{\text{st}}$ agent (media) to set of agents kN, which then takes the form $kN = \{v_1, \ldots v_{kn}, v_{media}\}$. Further perturb matrix W' to W'^* and preference profile R'_{-s} to $R'^*_{-s} \quad \forall s \in kN$, and then apply part 1. □

Theorem 3.6 (Higher manipulability of larger (extended) social groups). *Let $N = \{v_1, \ldots v_n\}$ be the initial set of agents and $K = \{k_1, \ldots k_L\}$ be the set of policies, let W be an $n \times n$ influence matrix and $\{(R_1^{(t)}, \ldots R_n^{(t)}) : t \in \mathbb{N} \cup \{0\}$ - path of the preference profiles.*

Let $k \in \mathbb{N}$ and $W' = \mathbf{I}_k \otimes W$ - matrix of the extended social group, that has k times more agents than initial group of agents, specifically $kN = \{v_1, \ldots v_{nk}\}$ and let $(R_1^{(t)}, \ldots, R_{kn}^{(t)})$ be the *corresponding preference profile with* $R^{(t)}_{mn+s} = R^{(t)}_s$ *for* $m \in \{0, \ldots, k-1\}$ *and* $s \in \{1, \ldots, n\}$ $\quad \forall t \in \mathbb{N}$ *(this idea stands for the fact, that two groups: larger and smaller one should have the same structure to make the degrees of their manipulabilities comparable).*

Let the preference profiles $R_{-s}, R'_{-(mn+s)}, R^*_{-s}, R'^*_{-(mn+s)}$ *be those defined in Proposition 3.5*

Then $manip(R^{(t)}_1, \ldots, R^{(t)}_{kn}) \geq manip(R^{(t)}_1, \ldots, R^{(t)}_n) \quad \forall t \in \mathbb{N}$ *.*

Proof. Fix $t \in \mathbb{N}$. For simplicity $\forall s$ instead of $R^{(t)}_s$ denote the preference of agent s as R_s, which is reasonable since t is fixed.

Pick $R_{med} \in 2^{K^2}$ such that $R_{med} = \underset{R \in 2^{K^2}}{\text{argmax}} \frac{1}{n} \sum_{s=1}^{n} d(f(R_s, R_{-s}, W), f(R_s, R^*_{-s}, W^*))$. Then we have:

$$manip(R_1, \ldots, R_{kn}) \overset{\text{def. 3.6 manipul. pref. profile}}{\geq} manip\ (R_{med}) \text{ under the pref. profile } (R_1, \ldots R_{kn}) =$$

$$\overset{\text{def 3.5 manipul. media signal}}{=} \frac{1}{kn} \sum_{s=1}^{kn} d\left(f(R_s, R'_{-s}, W'), f(R_s, R'^*_{-s}, W'^*)\right) =$$

$$= \frac{1}{k} \cdot \frac{1}{n} \cdot \left(\sum_{s=1}^{n} d\left(f(R_s, R'_{-s}, W'), f(R_s, R'^*_{-s}, W'^*)\right) + \ldots + \sum_{s=(k-1)n+1}^{kn} d\left(f(R_s, R'_{-s}, W'), f(R_s, R'^*_{-s}, W'^*)\right) \right) =$$

$$\overset{\text{Proposition 3.5}}{=} \frac{1}{k} \cdot \frac{1}{n} \cdot k \cdot \left(\sum_{s=1}^{n} d\left(f(R_s, R_{-s}, W), f(R_s, R^*_{-s}, W^*)\right) \right) = \frac{1}{n} \cdot \left(\sum_{s=1}^{n} d\left(f(R_s, R_{-s}, W), f(R_s, R^*_{-s}, W^*)\right) \right) =$$

$$\overset{\text{def 3.5 manipul. media signal}}{=} manip(R_{med}) \text{ under the preference profile } (R_1, \ldots R_n) \overset{\text{by construction of } R_{med} \text{ and by def. 3.6}}{=}$$

$$= manip(R_1, \ldots, R_n).$$

In other words, the statement above formally proves the following idea: for the behavioural mechanism and the group structures considered in this final project, the preference profiles of larger groups are (weakly) more manipulable by the media than those of smaller ones ceteris paribus.

□

4 Quantitative Examples

In this Section we present five examples of social groups, initial preference profiles and dynamics of preference update process. Specifically, we discuss the following cases:

- Society with fifteen interacting agents
- Society, with a group structure , that satisfies Sufficient Equilibrium conditions
- Society, which preference update dynamics exhibits inverse relation between diversity and manipulability of preference profiles
- Society with a groups structure, that leads to non-existence of equilibrium during the preference update process
- Societies with smaller and larger group sizes, but the same group structure

The coding part of this project, that produced graphs depicted below, can be found here.

4.1 Society with fifteen interacting individuals

Case 1: No Media Intervention

Assume we have $n = 15$ agents and $K = 3$ alternatives our agents have their preferences over. The structure of the links in the society is represented by the following 15 by 15 weighting (influence) matrix:

$$
W = \begin{bmatrix}
0.1 & 0.2 & 0.1 & 0.3 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.2 & 0.1 & 0.3 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.2 & 0.1 & 0.3 & 0.3 \\
0.2 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.1 & 0.5 \\
0.0 & 0.0 & 0.1 & 0.0 & 0.2 & 0.2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.5 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\
0.1 & 0.0 & 0.0 & 0.1 & 0.2 & 0.1 & 0.0 & 0.2 & 0.0 & 0.0 & 0.1 & 0.0 & 0.0 & 0.0 & 0.2 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.0 & 0.1 & 0.1 & 0.1 & 0.0 & 0.1 & 0.1 & 0.1 & 0.3 \\
0.2 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.1 & 0.1 & 0.1 & 0.3 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\
0.0 & 0.1 & 0.2 & 0.1 & 0.3 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.2 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \\
0.1 & 0.2 & 0.1 & 0.3 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0
\end{bmatrix}
$$

In this setting, media does not intervene in the updating process. It can be observed that both diversity and manipulability of preference profiles decrease along the equilibrium path until both of them eventually reach the value 0, which signifies a unanimous conclusion and thus non-manipulability.

Below are the graphical representations of preference profile diversity and manipulability dynamics:

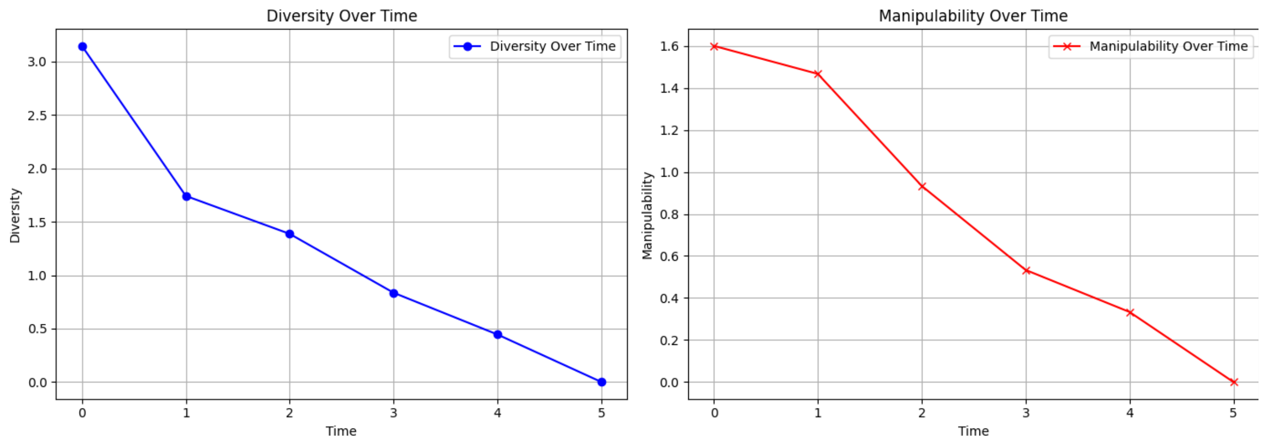

Figure 4.1: Preference profile diversity and manipulability dynamics along the equilibrium path under the assumption of no media shock for the society of 15 agents

At time $t = 0$, we have:

- **Preference Profile:**
  - $R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}; R_2 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}; R_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}; R_4 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}; R_5 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$
  - $R_6 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}; R_7 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}; R_8 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}; R_9 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}; R_{10} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$
  - $R_{11} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}; R_{12} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}; R_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}; R_{14} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}; R_{15} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
- **Number of Agents:** 15
- **Diversity:** 23.6
- **Manipulability:** 1.6

At time $t = 5$, we have:

- **Preference Profile:**
  - $R_s = R_r = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ for all $r, s \in \mathbb{N}$, which is the unanimous equilibrium.
- **Number of Agents:** 15
- **Diversity:** 0.0
- **Manipulability:** 0.0

Case 2: Media Intervention at Time 3 In the context of the same model, introduce the scenario where media intervention occurs at time point 3, assuming the role of a "spoiler." The media strategically transmits the most manipulative binary relations to the agents, specifically those that induce the maximum number of ranking changes across all individuals within a single iteration. It is noted that multiple preferences fit this criterion; for illustrative purposes, we select one such preference signal:

$R_{med} = \underset{R_{med} \in 2^{K^2}}{argmax} \sum_{r=1}^{N} d(f(R_r, R_{-r}, W), f(R_r, (R_1, ..., R_{r-1}, R_{r+1}, ..., R_{med}), W^*))$

Which in this case is equal to

$$R_{med} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

We can observe that the diversity and manipulability temporarily increase as a result of a media shock, but later, when the media shock is over, the agents continue their discussion and come to equilibrium, which is still very close to the one without media (Kemeny distance is just 1).

$$R_1^{(e)} = \ldots = R_{15}^{(e)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The graphical representation of diversity and manipulability dynamics is also provided below:

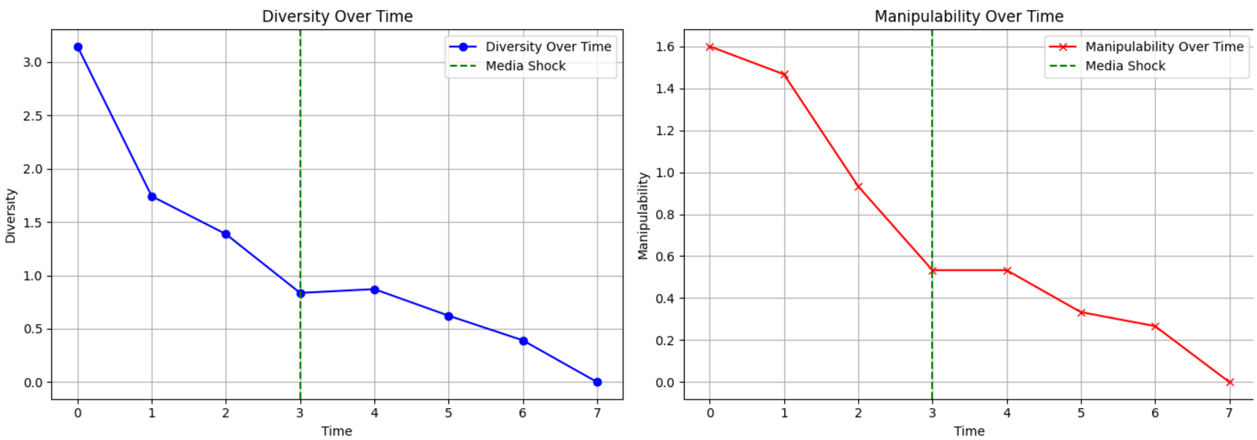

Figure 4.2: Preference profile diversity and manipulability dynamics along the equilibrium path under the media intervention with the "spoiler" media narrative at time $t = 3$ for the society of 15 agents

Case 3: Media Intervention at Time 3, but with Intention to Decrease Diversity

Above, we observed that media temporarily increases diversity, but over time, this effect disappears, although the equilibrium preference slightly differs. A natural question arises: can we decrease diversity through media intervention even more than it would have decreased if there were no media intervention? The answer in this case is "yes." For instance, one can consider the intervention (again at time point $t = 3$, for comparability) with another signal:

$$R_{med} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Then we can observe that both diversity and manipulability react with a more drastic decrease just after the media shock. Eventually, agents reach a unanimous equilibrium:

$$R_r^{(t)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \forall s \in N$$

Note that it is the same as if there were no media intervention. So in this case, media only changes dynamics, but does not change the equilibrium outcome.

The graphs in this case are the following:

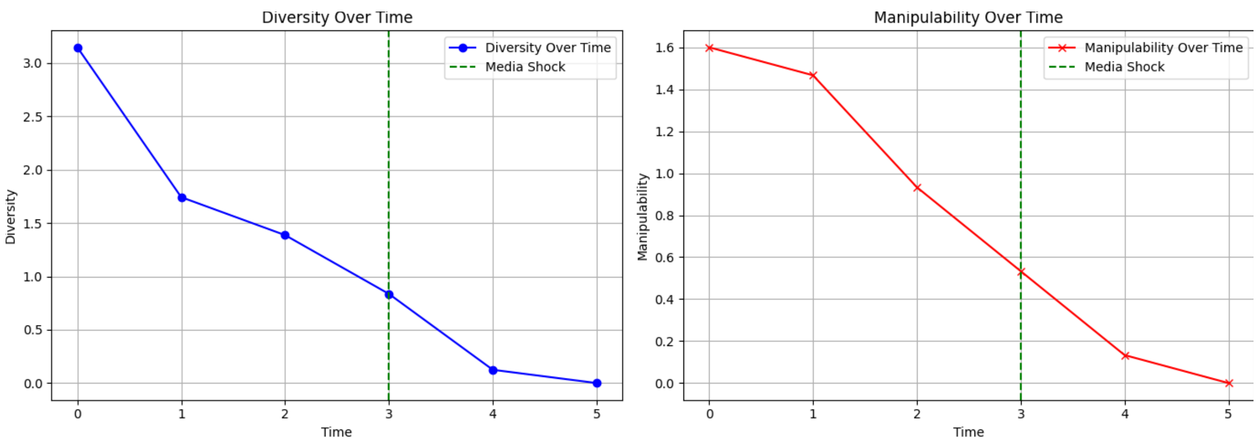

Figure 4.3: Preference profile diversity and manipulability dynamics along the equilibrium path under the media intervention with the diversity decreasing media narrative at time $t = 3$ for the society of 15 agents

Case 4: Media Intervention at Time 0

Notice that in the interventions above, the media intervened at time $t = 3$, when the manipulability of preferences was able to decrease during the preceding 3 periods ($t = 0, 1, 2$) of agents' discussions. Let us check if we are able to manipulate the outcome more when we are at the beginning of the equilibrium path, when the manipulability is larger.

In this case, the "spoiler" media narrative was calculated to be:

$$R_{med} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Then the equilibrium is again unanimous, but in this case it is "more" different from the equilibrium reached without media intervention than the one which is reached when media enters at time $t = 3$ (Kemeny distance between these equilibria is 3 instead of 1). So the equilibrium is:

$$R_1^{(e)} = \ldots = R_{15}^{(e)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

We can notice that the media play a coordinating effect: it changes people's opinions firstly, it changes them in the same way (which pushes diversity down from period 0 to period 1 when the media intervenes), which leads to the result that agents reach equilibrium faster (4 steps instead of 5 as in no-media Case 1), and also the new equilibrium is different from the old one (the Kemeny distance between them is 5).

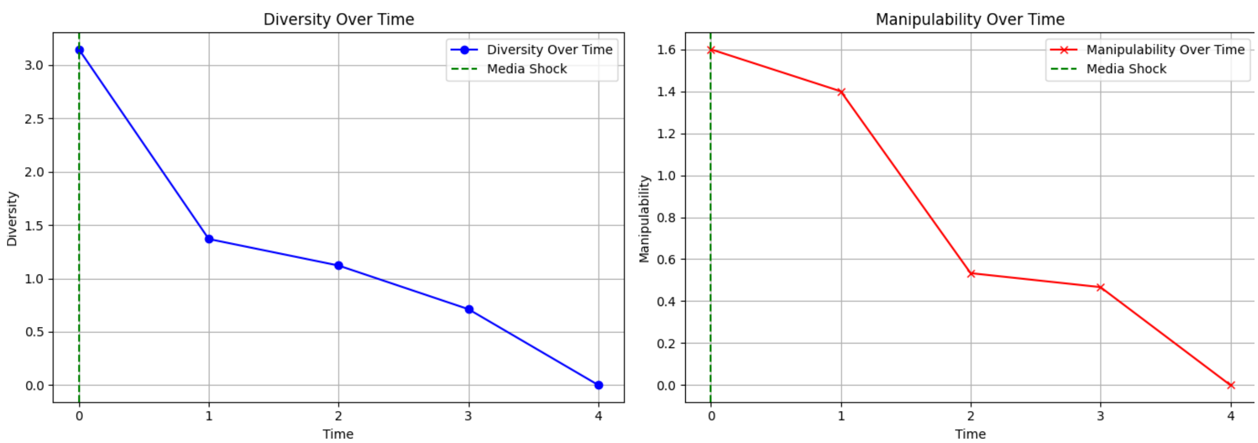

Figure 4.4: Preference profile diversity and manipulability dynamics along the equilibrium path under the media intervention with the "spoiler" narrative at time $t = 0$ for the society of 15 agents

Conclusions for Further example

One can conclude that our example demonstrates the following patterns:

- The equilibrium can be reached either faster or slower depending on the media signal and the time of intervention. Earlier media intervention leads to quicker equilibrium attainment (subject to the chosen media signal), as agents exhibit higher manipulability at earlier time points. Thus, media can steer the equilibrium in a specific "direction." Simultaneously, media could opt for "spoiler" signals that transiently increase diversity. Following the media shock, agents revert to discussions and reach a subsequent equilibrium.

- The equilibrium may differ. Transmitting the media signal earlier results in heightened manipulability within the preference profile, potentially leading to a larger Kemeny distance (again, contingent on the chosen signal) between the equilibrium without media intervention and the media-induced equilibrium.

In all cases observed, agents converge to a unanimous conclusion. However, depending on the signal transmitted by the media (if any) and the manipulability of agents at the point of media signal transmission:

4.2 Society structure that demonstrates sufficient equilibrium conditions

Case 1: No Media Intervention

To support the proof of Theorem 3.4 (Sufficient Conditions for Equilibrium Existence) by example, we have constructed the following society. We have $n = 9$ agents and $K = 1$ alternative (agents discuss whether they like or dislike the policy being proposed). It is meaningful to discuss only one alternative because our behavioral mechanism satisfies the assumption of independence of irrelevant alternatives. The agents have the following lower triangular 9×9 weighting (influence) matrix:

$$W = \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.60 & 0.40 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.40 & 0.30 & 0.30 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.25 & 0.25 & 0.25 & 0.25 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.20 & 0.20 & 0.09 & 0.16 & 0.35 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.10 & 0.10 & 0.10 & 0.10 & 0.25 & 0.35 & 0.00 & 0.00 & 0.00 \\ 0.07 & 0.07 & 0.07 & 0.07 & 0.07 & 0.25 & 0.40 & 0.00 & 0.00 \\ 0.05 & 0.05 & 0.05 & 0.15 & 0.05 & 0.05 & 0.25 & 0.35 & 0.00 \\ 0.07 & 0.07 & 0.07 & 0.07 & 0.07 & 0.07 & 0.07 & 0.07 & 0.44 \end{bmatrix}$$

Assuming that the initial preference profile is such that the first agent "likes" the alternative offered, while all other eight agents do not, we can demonstrate an extreme case of the dynamic process in this example. Specifically, it takes $n - 1$ steps (in this case $9 - 1 = 8$) to reach the equilibrium, as suggested by Theorem 3.4.

During the process of preference updates, agents 2, 3, and 4 sequentially adopt position "1" regarding the question if they like the policy being proposed, thereby increasing diversity. Initially, the leader was in the minority, but subsequently, other agents began to follow the leader's position. As a few agents follow, diversity increases; however, beyond a certain threshold, diversity starts to decrease again, since each of the agents from 2 to 9 eventually adopts the new (and the same) preference.

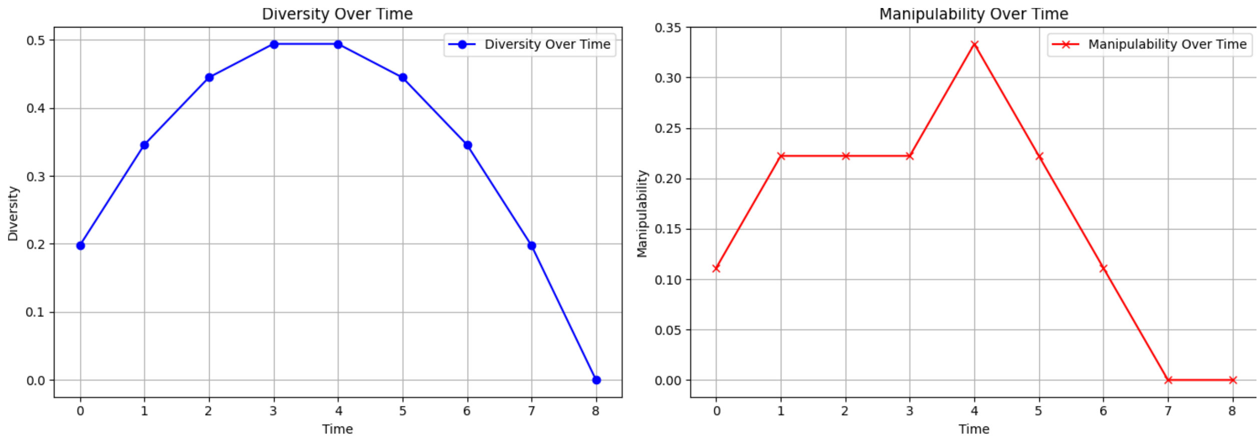

Figure 4.5: Preference profile diversity and manipulability dynamics along the equilibrium path under the assumption of no media intervention for society, which influence matrix satisfies the Sufficient equilibrium conditions

At the initial point in time $t = 0$, we have:

- **Time:** 0
- **Preference Profile:** $(R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_9) = (1, 0, 0, 0, 0, 0, 0, 0, 0)$
- **Number of Agents:** 9
- **Diversity:** 0.8888888888888888
- **Manipulability:** 0.1111111111111111

When the system reaches the equilibrium, we have:

- **Time:** 8
- **Preference Profile:** $(R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_9) = (1, 1, 1, 1, 1, 1, 1, 1, 1)$
- **Number of Agents:** 9
- **Diversity:** 0.0
- **Manipulability:** 0.0

Case 2: Media Intervention at Time 4

Notice that from the example in the previous case, we can infer that the highest manipulability is reached at time $t = 4$. Assume that the media intervenes at this time $t = 4$. Further assume that the goal of the media is to help agents reach equilibrium faster, so it transmits a signal $R_{\text{med}} = 1$ at time point $t = 4$. As a result, the outcomes are depicted below. Specifically, after the media influence, it takes only one step to reach equilibrium instead of the four steps it would have taken without media intervention.

The initial and final preference profiles are the same as in the Case 1. The only part that differs is the dynamics of the process.

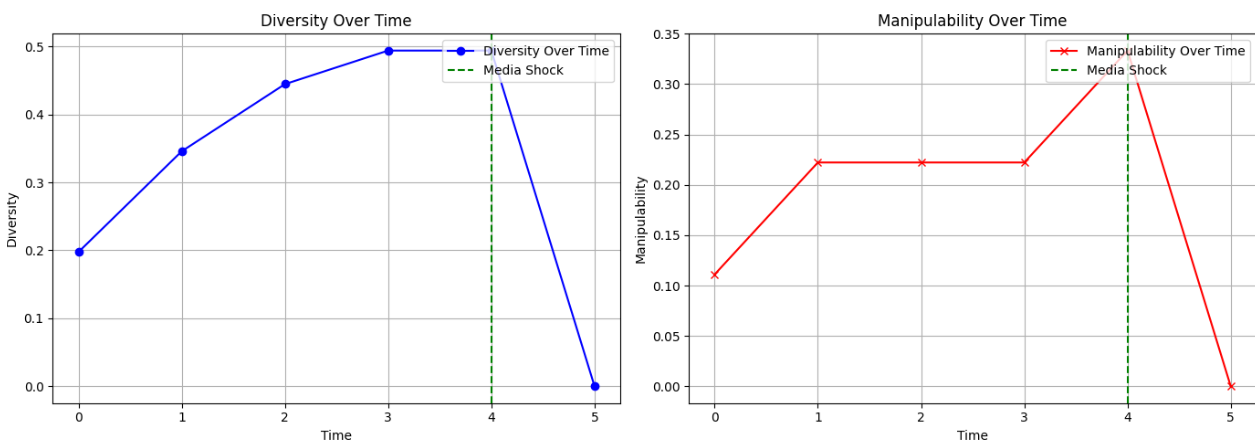

Figure 4.6: Preference profile diversity and manipulability dynamics along the equilibrium path under the under the media shock at time $t = 4$ for society, which influence matrix satisfies the Sufficient equilibrium conditions

Case 3: Media Intervention at Time 3

Now assume that the media intervenes at time point $t = 3$, when the manipulability is lower than at time $t = 4$, and communicates the same signal as before, $R_{\text{med}} = 1$.

In this case, it takes longer to reach the same equilibrium compared to the scenario when the media intervened at a time $t = 4$ of higher manipulability, but it is still faster compared to the case when there was no media intervention.

The initial and equilibrium preference profiles therefore remain the same as in the previous cases, but the dynamics change.

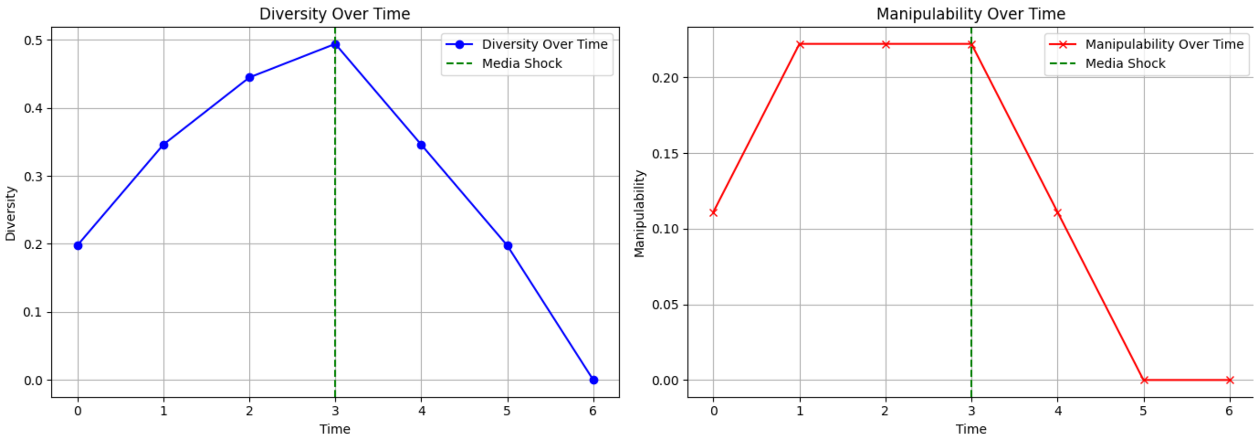

Figure 4.7: Preference profile diversity and manipulability dynamics along the equilibrium path under the under the media shock at time $t = 3$ for society, which influence matrix satisfies the Sufficient equilibrium conditions

Conclusion for this example

Therefore, we have demonstrated an example of a society with $n = 9$ agents and $K = 1$ policy, which each agent either likes or dislikes, and a lower triangular matrix W that, according to Theorem 3.4, guarantees the existence of equilibrium. Furthermore, our example illustrates the case where it takes the longest possible time to reach equilibrium, as suggested by the theorem. We have also influenced this system through media intervention, first at the time point $t = 4$ of highest possible manipulability, and then at time point $t = 3$ of lower manipulability.

We have found that, although the equilibrium preference profile remains unchanged, the media signal alters the equilibrium path and the time it takes to reach equilibrium. We have also noticed, that for this network structures increase in diversity is associated with the increase in manipulability. This has an interpretation that as agents have different positions, it gets easier for the media to manipulate them, because there are more chances that the media voice will be decisive. On the other hand, when the agents' position are of low diversity it gets more complicated for the media (if possible at all) since the majority of the weights supports the specific position, and the narrative of the media cannot change that.

4.3 Society with diversity and manipulability being inversely related for the first two time periods

Case: No media intervention

Assume we have $n = 7$ agents and $K = 5$ alternatives the agents have their preferences over. The structure of the society is represented by the following 7 by 7 weighting (influence) matrix

$$W = \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.60 & 0.40 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.40 & 0.30 & 0.30 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.25 & 0.25 & 0.25 & 0.25 & 0.00 & 0.00 & 0.00 \\ 0.20 & 0.20 & 0.10 & 0.30 & 0.20 & 0.00 & 0.00 \\ 0.10 & 0.10 & 0.10 & 0.10 & 0.35 & 0.25 & 0.00 \\ 0.07 & 0.07 & 0.07 & 0.07 & 0.07 & 0.25 & 0.40 \end{bmatrix}$$

And the initial preference profile:

- $R_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} R_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} R_3 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} R_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

- $R_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $R_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $R_7 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

In this setting, media does not intervene in the process. It can be observed that diversity decreases from periods $t = 0$ to $t = 1$, then increases from $t = 1$ to $t = 2$, stays constant from period $t = 2$ to $t = 3$, and then decreases up to period $t = 6$. Remarkably, the dynamics of diversity and manipulability are different between periods $t = 0$ and $t = 2$: whenever one metric increases, the other decreases, and vice versa.

This example demonstrates the idea that even though typically an increase in diversity is associated with an increase in manipulability and vice versa, under specific group structures and initial preference profiles (like those offered in this example) and specific time points along the equilibrium path, it is still possible that the diversity and manipulability of the preference profile are inversely related.

Below is the graph that demonstrates the communicated idea:

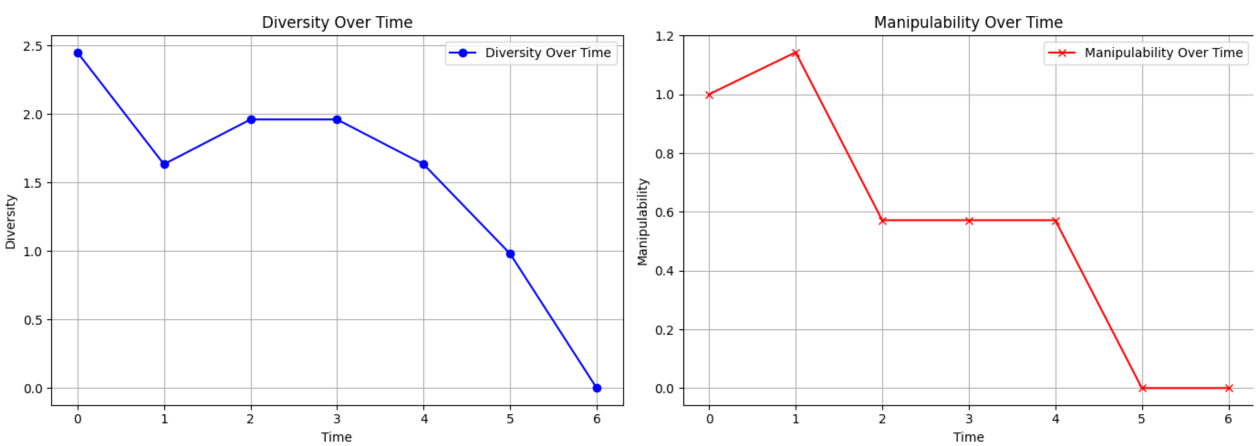

Figure 4.8: Preference profile diversity and manipulability dynamics along the equilibrium path when diveristy and manipulability are inversely related during periods 0 to 2 under the under the assumption of no media shock for the society of $n = 7$ agents

4.4 Society with non-existence of equilibrium

No media interention

Assume we have $n = 10$ agents and $K = 3$ alternatives the agents have their preferences over. The structure of the society is represented by the following 10 by 10 weighting (influence) matrix

$$W = \begin{bmatrix} 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.25 & 0.25 & 0.25 & 0.25 \\ 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.50 & 0.50 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.50 & 0.50 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.50 & 0.50 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.33 & 0.34 & 0.33 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.33 & 0.34 & 0.33 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.33 & 0.34 & 0.33 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.33 & 0.34 & 0.33 & 0.00 & 0.00 & 0.00 & 0.00 \end{bmatrix}$$

And the following initial preference profile:

- $R_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$; $R_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$;
- $R_6 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $R_8 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $R_9 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then as depicted on graphs below the preference profile path will oscillate, and the equilibrium will not exist.

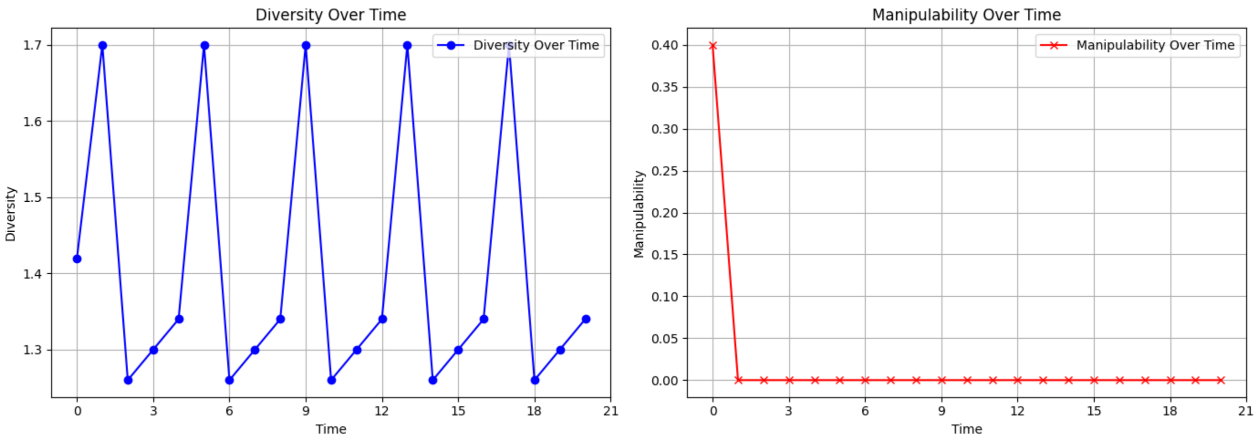

Figure 4.9: Preference profile diversity and manipulability dynamics along the oscillating path for the society of $n = 10$ agents

Remarkably, the manipulability of preference profiles decreases to 0 at time point $t = 1$,and does not increase in the upcoming periods. This suggests, that since manipulability gets equal to zero there is no media signal, which could redirect the preference profiles to the equilibrium path.

4.5 Societies with same form of influence links, but different group sizes

No media intervention

Assume we have $n = 4$ agents and $K = 5$ alternatives the agents have their preferences over. The structure of the society is represented by the following 4 by 4 weighting (influence) matrix

$$W = \begin{bmatrix} 0.15 & 0.35 & 0.15 & 0.35 \\ 0.20 & 0.30 & 0.40 & 0.10 \\ 0.40 & 0.00 & 0.30 & 0.30 \\ 0.20 & 0.30 & 0.00 & 0.50 \end{bmatrix}$$

The initial preference profile is given by

- $R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} R_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} R_3 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} R_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

The the dynamics of diversity and manipulability is calculated to be as follows:

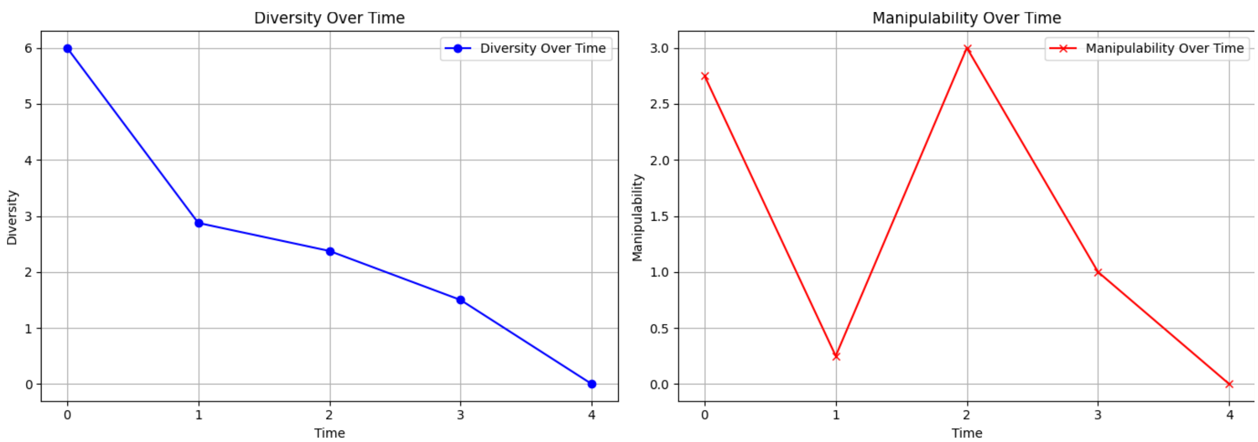

Figure 4.10: Preference profile diversity and manipulability dynamics along the equilibrium path for the society of $n = 4$ agents

Further, we will increase the group size without changing the group structure. Specifically, let $n = 8$ and $K = 5$. The influence matrix W for this larger group is given as follows:

$$W' = I_2 \otimes W = \begin{bmatrix} 0.15 & 0.35 & 0.15 & 0.35 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.20 & 0.30 & 0.40 & 0.10 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.40 & 0.00 & 0.30 & 0.30 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.20 & 0.30 & 0.00 & 0.50 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.15 & 0.35 & 0.15 & 0.35 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.20 & 0.30 & 0.40 & 0.10 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.40 & 0.00 & 0.30 & 0.30 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.20 & 0.30 & 0.00 & 0.50 \end{bmatrix}$$

The initial preference profile will be

- $R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} R_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} R_3 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} R_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$
- $R_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} R_6 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} R_7 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} R_8 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

Then the dynamics of diversity and manipulability is given below:

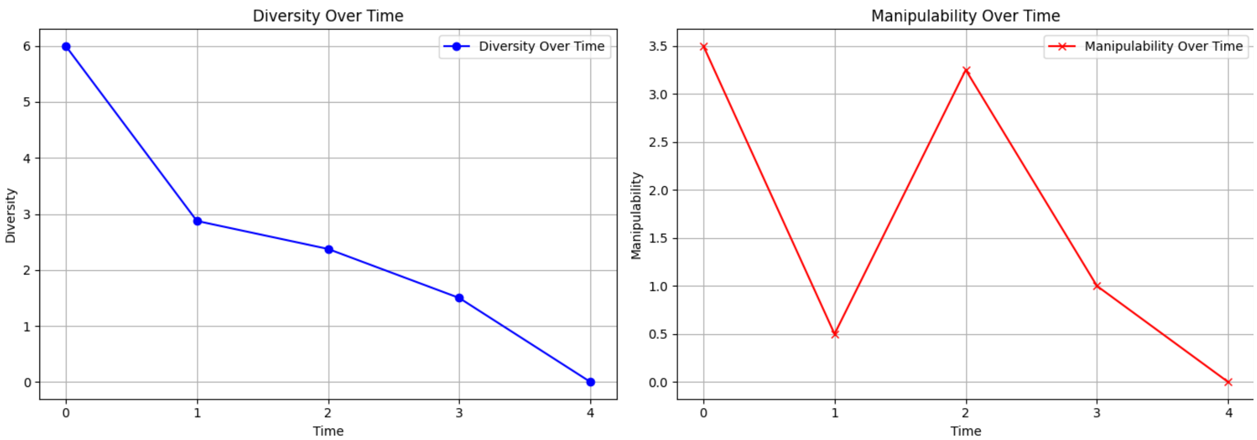

Figure 4.11: Preference profile diversity and manipulability dynamics along the equilibrium path for the society of $n = 8$ agents

One can notice, that diversity did not change, however the manipulability weakly increased: specifically it increased in time periods $t = 0, t = 1, t = 2$ and it stayed the same in periods $t = 3$ and $t = 4$, which should be the case according to Theorem 3.6, that we have proven in this Final project. Apart for considering figures 4.10 and 4.11 separately one can also observe the same comparison of manipulabilities of larger and smaller groups on the same figure.

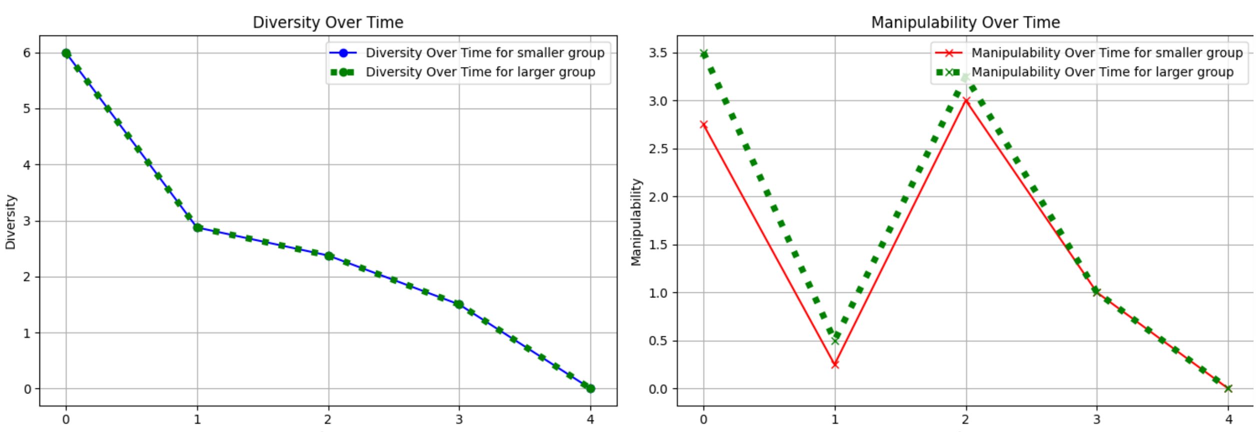

Figure 4.12: Preference profile diversity and manipulability dynamics along the equilibrium path for the societies of the same structure containing $n = 4$ and $n = 8$ agents

5 Conclusion

In this final project, we developed a model to describe how agents update their preferences over a set of policies as a result of interactions with one another. We formulated sufficient conditions for equilibrium and examined the relationship between the diversity of group preferences and their manipulability at each point along the equilibrium path.

Key findings include:

- **Existence of Equilibrium**: We found that a group structure represented by a lower triangular matrix, regardless of the initial preferences shared by the agents, always ensures the existence of equilibrium.
- **Preservation of completeness property of relations by the behavioural mechanism**: We found that a completeness property is kept and transitivity property is not kept by the proposed behavioural mechanism $f(\cdot)$.
- **Diversity and Manipulability**: Empirically, we observed that the diversity of preferences and the degree of their manipulability are often directly related. Higher diversity corresponds to greater ease for the media to influence positions, while lower diversity makes it more difficult (if not impossible) for the media to effect change. However, we also acknowledge the possibility of inverse relationships between diversity and manipulability, as well as scenarios where equilibrium does not exist and the preference update process leads to oscillations.
- **Computer Simulation**: We developed a computer simulation to model the equilibrium path if equilibrium exists, or the oscillating path if it does not. This simulation calculates and plots the dynamics of diversity and manipulability of a preference profile at each time point along the path.

- **Group Size and Media Influence**: We found that preferences of representatives of larger social groups are more manipulable by the media compared to those of smaller groups. Regardless of group size, each step towards equilibrium typically reduces diversity, making individual preferences less manipulable as equilibrium approaches. However, this does not preclude polarization, especially at the beginning of the updating process.

- **Media Shocks**: We examined the dynamics of the system when perturbed by media shocks. Our simulations showed that imposing a media shock at a point along the equilibrium path where the degree of manipulability is higher can lead to greater deviation from the no-intervention equilibrium (in terms of Kemeny distance) or alter the speed at which equilibrium is reached, depending on the media signal and the timing of the intervention.

In summary, our preference update mechanism and group structure indicate that the effect of media (manipulability of preferences) increases with group size, *ceteris paribus*. The lower triangular matrix ensures the existence of equilibrium, and our mechanism maintains the completeness of preference relations throughout the updating process. Empirically, we have shown that manipulability typically weakens as diversity in preferences decreases, provided all other parameters remain constant. However, further theoretical discussion is needed to fully understand the conditions under which these results hold.

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Acknowledgement

I would like to thank to Professor Hervé Crès for his support, guidance, and patience during the completion of this final project. Additionally, I am grateful to Professor Samreen Malik for her systematic approach and mentorship throughout this course, as well as for the valuable feedbacks provided.